

# Algebro-Geometric approach for a centrally extended $U_q[sl(2/2)]$ R-matrix

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## Abstract

In this paper we investigate the algebraic geometric nature of a solution of the Yang-Baxter equation based on the quantum deformation of the centrally extended  $sl(2|2)$  superalgebra proposed by Beisert and Koroteev [1]. We derive an alternative representation for the R-matrix in which the matrix elements are given in terms of rational functions depending on weights sited on a degree six surface. For generic gauge the weights geometry are governed by a genus one ruled surface while for a symmetric gauge choice the weights lie instead on a genus five curve. We have written down the polynomial identities satisfied by the R-matrix entries needed to uncover the corresponding geometric properties. For arbitrary gauge the R-matrix geometry is argued to be birational to the direct product  $\mathbb{CP}^1 \times \mathbb{CP}^1 \times A$  where  $A$  is an Abelian surface. For the symmetric gauge we present evidences that the geometric content is that of a surface of general type lying on the so-called Severi line with irregularity two and geometric genus nine. We discuss potential geometric degenerations when the two free couplings are restricted to certain one-dimensional subspaces.

Keywords: Yang-Baxter, R-matrix, Algebraic Geometry

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# 1 Introduction

A large variety of two-dimensional statistical mechanical models are known to be soluble on the basis of commuting transfer matrices method devised in the early 1970's by Baxter [2]. A given lattice model is described by its elementary Boltzmann weights  $\mathbf{w}$  which can be organized as points of  $n$ -dimensional projective space  $\mathbb{CP}^n$ . This means that the weights can be seen as set of  $n + 1$  coordinates  $[\omega_0 : \omega_1 : \dots : \omega_n]$  such that  $[\lambda\omega_0 : \lambda\omega_1 : \dots : \lambda\omega_n]$  are identified with  $[\omega_0 : \omega_1 : \dots : \omega_n]$  for any non zero number  $\lambda$ . Let  $T_N(\mathbf{w})$  be the transfer matrix of a given model which depends on the size  $N$  of the lattice. Baxter's approach to integrability assumes that it is possible to imbed the transfer matrix into a family of pairwise commuting operators,

$$[T_N(\mathbf{w}_1), T_N(\mathbf{w}_2)] = \mathbf{0} \quad \forall \mathbf{w}_1 \text{ and } \mathbf{w}_2. \quad (1)$$

For each value of  $N$  the condition (1) will lead us to a system of algebraic relations for the unknown weights  $\mathbf{w}$ . We then hope that after certain size  $N_0$  these polynomial constraints will become redundant in the sense that they will belong to the ideal generated by the relations coming from previous lattice sizes  $N < N_0$ . We next have to be able to recast these finite collection of algebraic equations in the form  $H_\alpha(\mathbf{w}_1)G_\alpha(\mathbf{w}_2) - G_\alpha(\mathbf{w}_1)H_\alpha(\mathbf{w}_2)$  where  $H_\alpha(\mathbf{w})$  and  $G_\alpha(\mathbf{w})$  are homogeneous polynomials with the same degree. In this situation, the weights will be sited in an algebraic variety  $X$  in  $\mathbb{CP}^n$  defined formally as,

$$X = \{\mathbf{w} \in \mathbb{CP}^n \mid P_1(\omega_0, \dots, \omega_n) = P_2(\omega_0, \dots, \omega_n) = \dots = P_k(\omega_0, \dots, \omega_n) = 0\}, \quad (2)$$

where  $P_\alpha(\omega_0, \dots, \omega_n) = H_\alpha(\omega_0, \dots, \omega_n) - \Lambda_\alpha G_\alpha(\omega_0, \dots, \omega_n)$  such that  $\Lambda_\alpha$  are coupling parameters.

In fact, Baxter has introduced a finite number of local conditions which are sufficient for the commutativity of two distinct transfer matrices. This is the celebrated Yang-Baxter algebra which for vertex models can be expressed in terms of product of matrices acting on three different spaces. Let us denote by  $L(\mathbf{w})$  the transition operator encoding the structure of the Boltzmann weights of the given vertex model. The transfer matrix can be written as ordered product of such operators and the commutativity condition (1) is assured provided that there exists an invertible matrix

$R(\mathbf{w}_1, \mathbf{w}_2)$  satisfying the algebraic relation,

$$R_{12}(\mathbf{w}_1, \mathbf{w}_2)L_{13}(\mathbf{w}_1)L_{23}(\mathbf{w}_2) = L_{23}(\mathbf{w}_2)L_{13}(\mathbf{w}_1)R_{12}(\mathbf{w}_1, \mathbf{w}_2), \quad (3)$$

where the subscript indices  $ij$  denote the two-dimensional subspace in which a given operator is acting on.

At this point we emphasize that the geometrical properties associated to the R-matrix can not in general be read directly from that of the Boltzmann weights. From the Yang-Baxter algebra the R-matrix elements can be retrieved by standard linear elimination and thus they define the following rational map,

$$\begin{array}{ccc} X \times X \subset \mathbb{CP}^n \times \mathbb{CP}^n & \xrightarrow{\phi(R)} & Y \subset \mathbb{CP}^m \\ [\mathbf{w}_1] \times [\mathbf{w}_2] & \longmapsto & [\phi_0(\mathbf{w}_1, \mathbf{w}_2), \dots, \phi_m(\mathbf{w}_1, \mathbf{w}_2)], \end{array}$$

where  $m$  refers to the number of linearly independent R-matrix elements and  $\phi_j(\mathbf{w}_1, \mathbf{w}_2)$  are bi-homogeneous polynomials on two distinct sets of weights.

The algebraic geometry of the R-matrix is then described by  $Y$  which does not need to coincide with the geometric properties of the product  $X \times X$  since generically  $\phi(R)$  can be a high degree map far from a birational equivalence<sup>1</sup>. In order to study the geometry of  $Y$  we need to determine its defining equations which are obtained by computing the implicit representation of the image of the rational map  $\phi(R)$ . This task is accomplished by eliminating the variables  $\mathbf{w}_1$  and  $\mathbf{w}_2$  out of the following ideal,

$$\mathbf{I} = \langle P_1(\mathbf{w}_1), \dots, P_k(\mathbf{w}_1); P_1(\mathbf{w}_2), \dots, P_k(\mathbf{w}_2); r_0 - \phi_0(\mathbf{w}_1, \mathbf{w}_2), \dots, r_m - \phi_m(\mathbf{w}_1, \mathbf{w}_2) \rangle, \quad (4)$$

where  $r_0, \dots, r_m$  denote the independent entries of the R-matrix.

As a result of the elimination procedure we will find a number of algebraic constraints among the R-matrix elements making it possible to formally represent the variety  $Y$  as,

$$Y = \{[r_0 : \dots : r_m] \in \mathbb{CP}^m \mid Q_1(r_0, \dots, r_m) = Q_2(r_0, \dots, r_m) = \dots = Q_l(r_0, \dots, r_m) = 0\}, \quad (5)$$

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<sup>1</sup> In the special cases where  $X$  and  $Y$  are the same varieties the rational map  $\phi(R)$  plays the role of an addition rule among the weights typical of algebraic groups.

where  $Q_\alpha(r_0, \dots, r_m)$  are yet another family of homogenous polynomials.

In principle, the explicit expressions for  $Q_\alpha(r_0, \dots, r_m)$  can be obtained by using an alternative representation for the ideal  $\mathbf{I}$  denominated Groebner bases [3]. However, in practice this task is algorithmically involved depending much on the complexity of the polynomials defining both  $X$  and the rational map  $\phi(R)$ . For an example in the case of the vertex model associated to the Hubbard chain we refer to [4].

Having at hand a solution of the Yang-Baxter equation it is of interest to uncover the geometric content of both varieties  $X$  and  $Y$ . This is specially relevant in situations where the elements of the  $R$ -matrix are not all expressed in terms of rational functions. The presence of multiple coverings such as square roots terms could hide the actual geometric content underlying the Yang-Baxter solution. In this paper we investigate this issue for a  $R$ -matrix based on a deformation of the centrally extended  $sl(2|2)$  superalgebra found by Beisert and Koroteev within the quantum group machinery [1]. For generic gauge we show that the Boltzmann weights sit on a surface ruled by an elliptic curve which has a degree two isogeny with the genus one parameterization devised by Beisert and Koroteev. It turns out that the suitable symmetric gauge choice made in [1] does not cut the ruled surface on its  $\mathbb{CP}^1$  fibre and in this case the Boltzmann weights lie on a curve of genus five. The geometric properties of the  $R$ -matrix for generic gauge are argued to be governed by the product variety  $\mathbb{CP}^1 \times \mathbb{CP}^1 \times A$  where  $A$  is an Abelian surface. However, for the symmetric calibration, we present strong evidences that the surface is of general type sitting on the Severi line [5], that is, the canonical class  $\mathcal{K}_S$  and the Euler-Poincaré characteristic  $\chi(S)$  of the surface satisfy the relation  $\mathcal{K}_S^2 = 4\chi(S)$ . These results generalize in a substantial way the recent work [4] associated to the specific undeformed case.

We have organized this paper as follows. In next section we derive an alternative representation for the  $R$ -matrix such that the matrix elements are rational functions of certain elementary weights sited on a degree six surface. This provides the basics to investigate the geometrical properties associated to both the Boltzmann weights and the  $R$ -matrix performed in sections 3. In section 4 we discuss the geometrical content in the interesting case of a symmetric gauge choice. Our conclusions are in section 5 and in three appendices we summarize some technical details omitted

in the main text.

## 2 The $q$ -deformed R-matrix

We start recalling that the four-dimensional representation of the quantum deformation of the extended  $sl(2|2)$  superalgebra has been parametrized in terms of three variables denoted by  $x^+, x^-$  and  $\gamma$ . The latter plays the role of a free gauge parameter while  $x^\pm$  are required to fulfill the following elliptic curve [1],

$$E_1 = \frac{x_+}{q} + \frac{q}{x_+} - qx_- - \frac{1}{qx_-} + ig(q - 1/q)\left(\frac{x_+}{qx_-} - \frac{qx_-}{x_+}\right) - \frac{i}{g}, \quad (6)$$

where  $q$  denotes the deformation parameter and  $g$  is a coupling constant.

The intertwining operator encoding the graded structure of such fundamental representation has been originally constructed by Beisert and Koroteev [1]. For the purposes of this paper it is enough to consider the related R-matrix satisfying the standard Yang-Baxter equation, namely

$$R_{12}(\mathbf{w}_1, \mathbf{w}_2)R_{13}(\mathbf{w}_1, \mathbf{w}_3)R_{23}(\mathbf{w}_2, \mathbf{w}_3) = R_{23}(\mathbf{w}_2, \mathbf{w}_3)R_{13}(\mathbf{w}_2, \mathbf{w}_3)R_{12}(\mathbf{w}_1, \mathbf{w}_2). \quad (7)$$

Following the original work [1] we can represent the operator R by the following matrix,

$$R = \left( \begin{array}{cccc|cccc|cccc|cccc} \mathcal{A} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{B} & 0 & 0 & \mathcal{C} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{B} & 0 & 0 & 0 & 0 & 0 & \mathcal{C} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{F} & 0 & 0 & \frac{\mathcal{D}}{\delta} & 0 & 0 & \frac{-q\mathcal{D}}{\delta} & 0 & 0 & \mathcal{A} - q\mathcal{F} & 0 & 0 & 0 \\ \hline 0 & \overline{\mathcal{C}} & 0 & 0 & \overline{\mathcal{B}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta q \overline{\mathcal{D}} & 0 & 0 & \mathcal{G} & 0 & 0 & 1 - q\mathcal{G} & 0 & 0 & \delta q^2 \overline{\mathcal{D}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{\mathcal{B}} & 0 & 0 & 0 & 0 & 0 & \overline{\mathcal{C}} & 0 & 0 \\ \hline 0 & 0 & \overline{\mathcal{C}} & 0 & 0 & 0 & 0 & 0 & \overline{\mathcal{B}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta \overline{\mathcal{D}} & 0 & 0 & 1 - \frac{\mathcal{G}}{q} & 0 & 0 & \mathcal{G} & 0 & 0 & -\delta q \overline{\mathcal{D}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \overline{\mathcal{B}} & 0 & 0 & \overline{\mathcal{C}} & 0 \\ \hline 0 & 0 & 0 & \mathcal{A} - \frac{\mathcal{F}}{q} & 0 & 0 & \frac{-\mathcal{D}}{\delta q} & 0 & 0 & \frac{\mathcal{D}}{\delta} & 0 & 0 & \mathcal{F} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{C} & 0 & 0 & 0 & 0 & 0 & \mathcal{B} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{C} & 0 & 0 & \mathcal{B} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{A} \end{array} \right), \quad (8)$$

where  $\delta$  is a free twist parameter.

Defining the auxiliary parameter  $\xi = ig(q - 1/q)$  the matrix elements can be expressed as,

$$\begin{aligned}
\mathcal{A} &= \frac{(x_1^- - x_2^+) \sqrt{(\xi + x_1^+)(\xi + x_2^-)}}{(x_2^- - x_1^+) \sqrt{(\xi + x_2^+)(\xi + x_1^-)}}, \quad \mathcal{B} = \frac{(x_1^+ - x_2^+) \sqrt{\xi + x_2^-}}{\sqrt{q}(x_1^+ - x_2^-) \sqrt{\xi + x_2^+}}, \\
\overline{\mathcal{B}} &= \frac{\sqrt{q}(x_1^- - x_2^-) \sqrt{\xi + x_1^+}}{(x_1^+ - x_2^-) \sqrt{\xi + x_1^-}}, \quad \mathcal{C} = \frac{\gamma_2(x_1^- - x_1^+) \sqrt{(\xi + x_1^+)(\xi + x_2^-)}}{\gamma_1(x_2^- - x_1^+) \sqrt{(\xi + x_2^+)(\xi + x_1^-)}}, \\
\overline{\mathcal{C}} &= \frac{\gamma_1(x_2^- - x_2^+)}{\gamma_2(x_2^- - x_1^+)}, \quad \mathcal{D} = \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^+ - x_1^+) \sqrt{\xi + x_2^-}}{\gamma_1 \gamma_2 (x_2^- - x_1^+) \sqrt{\xi + x_2^+} [1 - \xi(x_1^- + x_2^-) - x_1^- x_2^-]}, \\
\overline{\mathcal{D}} &= \frac{\gamma_1 \gamma_2 (1 + \xi^2) (x_2^+ - x_1^+) (\xi + x_2^-) \sqrt{\xi + x_1^-}}{q^3 (x_2^- - x_1^+) (\xi + x_2^+) \sqrt{\xi + x_1^+} [1 - \xi(x_1^- + x_2^-) - x_1^- x_2^-]}, \\
\mathcal{F} &= \frac{(x_1^+ - x_2^+) \sqrt{(\xi + x_1^-)(\xi + x_2^-)} [1 - \xi(x_1^+ + x_2^-) - x_1^+ x_2^-]}{q(x_2^- - x_1^+) \sqrt{(\xi + x_1^+)(\xi + x_2^+)} [1 - \xi(x_1^- + x_2^-) - x_1^- x_2^-]}, \\
\mathcal{G} &= \frac{(\xi + x_2^-) (x_2^+ - x_1^+) [1 - \xi(x_1^- + x_2^+) - x_1^- x_2^+]}{q(\xi + x_2^+) (x_2^- - x_1^+) (1 - \xi(x_1^- + x_2^-) - x_1^- x_2^-)}, \tag{9}
\end{aligned}$$

where the subscript index  $j$  means distinct points  $x_j^\pm$  on the curve (6).

In this representation we see that many of the matrix elements (9) contain square root terms. This fact hides the actual geometric content associated to such solution of the Yang-Baxter equation (7). The ideal situation is to have all the R-matrix elements written only in terms of ratios of bi-homogeneous polynomials and a systematic way to uncover such algebraic structure is as follows. We first expand one of the R-matrix rapidities pair around a generic point belonging to the curve (6). The next step is to identify some of the expanded entries of the R-matrix with the coordinates  $[x : y : z : w]$  of a three-dimensional projective space. This leads to constraints which need to be solved for the variables  $x^\pm, \gamma$  and afterwards we should verify that all the matrix elements are indeed rational functions on the ring  $\mathbb{C}[x, y, z, w]$ . It turns out that one possible reference point is,

$$x^+ = -\xi + \epsilon, \quad x^- = \frac{1 + \xi^2}{q^2} \frac{1}{\epsilon}, \quad \gamma = \frac{1}{q^{1/4}} + \epsilon, \tag{10}$$

where  $\epsilon$  is an expansion variable.

We now expand the second set of rapidities of the R-matrix (8,9) around the above point and search for the simplest ratios among the expanded matrix entries. We find that these are given by the amplitudes  $\mathcal{A}/\mathcal{C}$ ,  $\mathcal{B}/\mathcal{C}$  and  $\bar{\mathcal{C}}/\mathcal{C}$  which in the limit  $\epsilon \rightarrow 0$  gives, respectively, the identification,

$$\frac{x}{w} = \frac{q^{1/4}\gamma(\xi + x^-)}{x^- - x^+}, \quad \frac{y}{w} = \frac{\gamma\sqrt{(\xi + x^-)(\xi + x^+)}}{q^{1/4}(x^+ - x^-)}, \quad \frac{z}{w} = \frac{\gamma^2\sqrt{(1 + \xi^2)(\xi + x^-)}}{q^{1/2}(x^- - x^+)\sqrt{\xi + x^+}}. \quad (11)$$

In order to solve the above constraints for the variables  $x^\pm$  and  $\gamma$  we consider the relations coming from the ratios  $x^2/y^2$  and  $y/z$ . This provides three linear equations which are easily solved and the final result is,

$$x_+ = -\xi - \frac{\sqrt{1 + \xi^2}}{\sqrt{q}} \left(\frac{y}{x}\right) \frac{(x^2 - qy^2)}{zw}, \quad x_- = -\xi - \frac{\sqrt{1 + \xi^2}}{q^{3/2}} \left(\frac{x}{y}\right) \frac{(x^2 - qy^2)}{zw}, \quad \gamma = \frac{x^2 - qy^2}{q^{1/4}xw}. \quad (12)$$

By using the above relations we have checked that matrix elements of the R-matrix expansion are indeed expressed in terms of ratios of polynomials on  $\mathbb{C}[x, y, z, w]$ . Note that relations (12) define a two-to-one mapping on the affine space  $w = 1$  and therefore the geometrical properties are not fully captured by the algebraic geometry description on the variables  $x^\pm$  and  $\gamma$ . The proper geometric content of the Boltzmann weights should be uncovered from the polynomial constraining the homogeneous variables  $x, y, z, w$  which is obtained by substituting Eq.(12) into the original elliptic curve (6). After some cumbersome simplifications we find that such algebraic surface is defined by,

$$S = (x^2 - \frac{y^2}{q})(x^2 - qy^2)^2 - Uxyzw(x^2 - qy^2) - w^2z^2(x^2 - q^3y^2), \quad (13)$$

where the Hubbard like coupling  $U$  is,

$$U = \frac{\sqrt{q}[1 - 2g^2(q - 1/q)^2]}{g\sqrt{g^2(q - 1/q)^2 - 1}}. \quad (14)$$

Finally, we present the expression for the R-matrix in terms of the surface  $S$  variables. As



before its basic matrix structure is given by,

$$R = \left( \begin{array}{cccc|cccc|cccc|cccc} \mathbf{a} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{b} & 0 & 0 & \mathbf{c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{b} & 0 & 0 & 0 & 0 & 0 & \mathbf{c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{f} & 0 & 0 & \frac{\mathbf{d}}{\delta_1} & 0 & 0 & -\frac{q\mathbf{d}}{\delta_1} & 0 & 0 & \mathbf{a} - q\mathbf{f} & 0 & 0 & 0 \\ \hline 0 & \bar{\mathbf{c}} & 0 & 0 & \bar{\mathbf{b}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{g} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q\delta_1\bar{\mathbf{d}} & 0 & 0 & \bar{\mathbf{g}} & 0 & 0 & \mathbf{g} - q\bar{\mathbf{g}} & 0 & 0 & q^2\delta_1\bar{\mathbf{d}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\mathbf{b}} & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\mathbf{c}} & 0 \\ \hline 0 & 0 & \bar{\mathbf{c}} & 0 & 0 & 0 & 0 & 0 & \bar{\mathbf{b}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta_1\bar{\mathbf{d}} & 0 & 0 & \mathbf{g} - \frac{\bar{\mathbf{g}}}{q} & 0 & 0 & \bar{\mathbf{g}} & 0 & 0 & -q\delta_1\bar{\mathbf{d}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{g} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\mathbf{b}} & 0 & 0 & \bar{\mathbf{c}} & 0 \\ \hline 0 & 0 & 0 & \mathbf{a} - \frac{\mathbf{f}}{q\delta_1} & 0 & 0 & -\frac{\mathbf{d}}{q\delta_1} & 0 & 0 & \frac{\mathbf{d}}{\delta_1} & 0 & 0 & \mathbf{f} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{c} & 0 & 0 & 0 & 0 & 0 & \mathbf{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{c} & 0 & 0 & \mathbf{b} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{a} \end{array} \right)$$

where the twist relation is  $\delta_1 = -\delta/q$ .

The matrix elements are obtained by using the mapping (12) on the previous amplitudes given

by Eq.(9). After some algebra and up to an overall normalization we obtain,

$$\begin{aligned}
\frac{\mathbf{a}}{\mathbf{c}} &= \frac{\mathbf{x}_1 \mathbf{x}_2}{\theta(\mathbf{x}_2, \mathbf{y}_2)} - q \frac{\mathbf{z}_1}{\mathbf{z}_2} \frac{\mathbf{y}_1 \mathbf{y}_2}{\theta(\mathbf{x}_1, \mathbf{y}_1)}, \quad \frac{\mathbf{b}}{\mathbf{c}} = \frac{\mathbf{y}_1 \mathbf{x}_2}{\theta(\mathbf{x}_2, \mathbf{y}_2)} - \frac{\mathbf{z}_1}{\mathbf{z}_2} \frac{\mathbf{x}_1 \mathbf{y}_2}{\theta(\mathbf{x}_1, \mathbf{y}_1)}, \quad \frac{\bar{\mathbf{c}}}{\mathbf{c}} = \frac{\mathbf{z}_1}{\mathbf{z}_2}, \\
\frac{\bar{\mathbf{b}}}{\mathbf{c}} &= q \frac{\mathbf{x}_1 \mathbf{y}_2}{\theta(\mathbf{x}_2, \mathbf{y}_2)} - q \frac{\mathbf{z}_1}{\mathbf{z}_2} \frac{\mathbf{y}_1 \mathbf{x}_2}{\theta(\mathbf{x}_1, \mathbf{y}_1)}, \quad \frac{\mathbf{g}}{\mathbf{c}} = \frac{\mathbf{z}_1}{\mathbf{z}_2} \frac{\mathbf{x}_1 \mathbf{x}_2}{\theta(\mathbf{x}_1, \mathbf{y}_1)} - q \frac{\mathbf{y}_1 \mathbf{y}_2}{\theta(\mathbf{x}_2, \mathbf{y}_2)}, \\
\frac{\mathbf{d}}{\mathbf{c}} &= \frac{\mathbf{x}_1 \mathbf{y}_1 \theta(\mathbf{x}_1, \mathbf{y}_1) (\mathbf{x}_2^2 - q^3 \mathbf{y}_2^2) - \frac{\mathbf{z}_1}{\mathbf{z}_2} \mathbf{x}_2 \mathbf{y}_2 \theta(\mathbf{x}_2, \mathbf{y}_2) (\mathbf{x}_1^2 - q^3 \mathbf{y}_1^2)}{\theta(\mathbf{x}_1, \mathbf{y}_1) \theta(\mathbf{x}_2, \mathbf{y}_2) (\mathbf{x}_1^2 \mathbf{x}_2^2 - q^2 \mathbf{y}_1^2 \mathbf{y}_2^2)}, \quad \frac{\bar{\mathbf{d}}}{\mathbf{c}} = \mathbf{z}_1 \mathbf{z}_2 \frac{\mathbf{d}}{\mathbf{c}}, \\
\frac{\mathbf{f}}{\mathbf{c}} &= \frac{\mathbf{x}_1 \mathbf{y}_1 \left[ \mathbf{x}_2 \mathbf{y}_1 \theta(\mathbf{x}_1, \mathbf{y}_1) - \frac{\mathbf{z}_1}{\mathbf{z}_2} \mathbf{x}_1 \mathbf{y}_2 \theta(\mathbf{x}_2, \mathbf{y}_2) \right]}{\theta(\mathbf{x}_1, \mathbf{y}_1) (\mathbf{x}_1^2 \mathbf{x}_2^2 - q^2 \mathbf{y}_1^2 \mathbf{y}_2^2)} + \frac{q^2 \mathbf{x}_2 \mathbf{y}_2 \left[ \mathbf{x}_1 \mathbf{y}_2 \theta(\mathbf{x}_1, \mathbf{y}_1) - \frac{\mathbf{z}_1}{\mathbf{z}_2} \mathbf{x}_2 \mathbf{y}_1 \theta(\mathbf{x}_2, \mathbf{y}_2) \right]}{\theta(\mathbf{x}_2, \mathbf{y}_2) (\mathbf{x}_1^2 \mathbf{x}_2^2 - q^2 \mathbf{y}_1^2 \mathbf{y}_2^2)}, \\
\frac{\bar{\mathbf{g}}}{\mathbf{c}} &= \frac{[q^2 \mathbf{z}_1 \mathbf{z}_2 \mathbf{x}_2 \mathbf{y}_1 - \mathbf{x}_1 \mathbf{y}_2 \theta(\mathbf{x}_1, \mathbf{y}_1) \theta(\mathbf{x}_2, \mathbf{y}_2)]}{\theta(\mathbf{x}_1, \mathbf{y}_1) \theta(\mathbf{x}_2, \mathbf{y}_2)} \frac{\mathbf{d}}{\mathbf{c}}, \tag{15}
\end{aligned}$$

where  $\theta(\mathbf{x}, \mathbf{y}) = \mathbf{x}^2 - q\mathbf{y}^2$  and the bold letters refer to the affine coordinates  $\mathbf{x} = x/w$ ,  $\mathbf{y} = y/w$ , and  $\mathbf{z} = z/w$ .

As expected the R-matrix are expressed solely in terms of ratios of bi-homogeneous polynomials on the coordinates  $\mathbf{x}_j$ ,  $\mathbf{y}_j$  and  $\mathbf{z}_j$ . At this point we have gathered the basic ingredients to study the geometrical properties of both the Boltzmann weights and the R-matrix.

### 3 Algebraic geometry for arbitrary gauge

We start by analyzing the geometrical properties of the Boltzmann weights associated to the sextic surface (13). This surface contains one-dimensional singularities which in principle should be resolved by means of birational morphisms. A partial desingularization is performed eliminating the product term  $zw$  by means of standard quadrature. This makes it possible to decrease the degree of the surface polynomial and as a result we have the following map,

$$\begin{aligned}
\mathbf{S} \subset \mathbb{CP}^3 & \xrightarrow{\quad \phi \quad} \widetilde{\mathbf{S}} \subset \mathbb{CP}^3 \\
[\mathbf{x}:\mathbf{y}:\mathbf{z}:\mathbf{w}] & \longmapsto \left[ \frac{\phi_1}{\phi_2} : w : x : y \right],
\end{aligned} \tag{16}$$

where the polynomials  $\phi_1 = \imath\sqrt{q}[U(x^2 - qy^2)xy + 2(x^2 - q^3y^2)zw]$  and  $\phi_2 = (x^2 - qy^2)w$  while  $\tilde{S}$  is a degree four surface defined by,

$$\tilde{S} = x_0^2x_1^2 + 4qx_2^4 - (4 - qU^2 + 4q^4)x_2^2x_3^2 + 4q^3x_3^4. \quad (17)$$

The above map defines a birational equivalence since it is invertible away from the singular locus of the surface  $S$ . The inverse map is given by,

$$\begin{aligned} \tilde{S} \subset \mathbb{CP}^3 & \xrightarrow{\phi^{-1}} S \subset \mathbb{CP}^3 \\ [x_0:x_1:x_2:x_3] & \longmapsto [x_2 : x_3 : \frac{\psi_2}{\psi_1} : x_1], \end{aligned} \quad (18)$$

where  $\psi_1 = 2\imath\sqrt{q}x_1(x_2^2 - q^3x_3^2)$  and  $\psi_2 = (x_0x_1 - \imath\sqrt{q}Ux_2x_3)(x_2^2 - qx_3^2)$ .

The next step is to observe that out of  $\tilde{S}$  one can define a surjective projection to an elliptic curve,

$$\tilde{S} \subset \mathbb{CP}^3 \xrightarrow{\pi} E_2 \subset \mathbb{CP}^2, \quad (19)$$

such that the fibre  $\pi^{-1}$  at every point on  $E_2$  is isomorphic to  $\mathbb{CP}^1$ . The affine form of  $E_2$  is that of a Jacobi quartic, namely

$$E_2 = y_1^2 + 4q - (4 - qU^2 + 4q^4)y_2^2 + 4q^3y_2^4. \quad (20)$$

Putting all these results together we conclude that  $S$  is in fact a surface ruled by a genus one curve, that is  $S \cong \mathbb{CP}^1 \times E_2$ . In order to provide a concrete representation for the surface  $S$  variables we associate the coordinate  $t$  to the affine part of its  $\mathbb{CP}^1$  subspace. Taking into account the form of the inverse map  $\phi^{-1}$  we can express the ratios of the surface  $S$  variables as follows,

$$\frac{x}{w} = t, \quad \frac{y}{w} = ty_2, \quad \frac{z}{w} = t^2 \frac{(y_1 - \imath\sqrt{q}Uy_2)(1 - qy_2^2)}{2\imath\sqrt{q}(1 - q^3y_2^2)}. \quad (21)$$

whose uniformization depends only on the curve  $E_2$ . In the appendix *A* we present one such uniformization for the variables  $y_1$  and  $y_2$ .

We next remark that the elliptic curves  $E_1$  and  $E_2$  are not isomorphic but only twofold isogenous. Note that the degree of the isogeny is in accordance with the map (12) degree. A simple

way to see this fact is through the comparison of their J-invariants since they will fix two points on the genus one curve moduli space [6]. In terms of the deformation parameter  $q$  and the coupling  $U$  the expressions of the corresponding J-invariants are,

$$\begin{aligned} J(E_1) &= \frac{(16 - 8qU^2 + q^2U^4 - 16q^4 - 8q^5U^2 + 16q^8)^3}{q^8(4 - qU^2 + 4q^4 + 8q^2)(4 - qU^2 + 4q^4 - 8q^2)} \\ J(E_2) &= \frac{(16 - 8qU^2 + q^2U^4 + 224q^4 - 8q^5U^2 + 16q^8)^3}{q^4(4 - qU^2 + 4q^4 + 8q^2)^2(4 - qU^2 + 4q^4 - 8q^2)^2}. \end{aligned} \quad (22)$$

Because the J-invariants are not the same the two curves can not be isomorphic for arbitrary values of the parameters  $q$  and  $U$ . They are however related by an isogeny of degree two and this feature can be verified with the help of the respective modular polynomial. This is a symmetric two variable polynomial with suitable coefficients and its explicit expression is,

$$\begin{aligned} \Phi_2[x, y] &= x^3 + y^3 - x^2y^2 + 1488xy(x + y) - 162000(x^2 + y^2) + 40773375xy \\ &+ 8748000000(x + y) - 157464000000000. \end{aligned} \quad (23)$$

Two elliptic curves are said to be twofold isogenous provided the valuation of the modular polynomial (23) at distinct J-invariants is zero. Taking into account the expressions (22) we find indeed that  $\Phi_2[J(E_1), J(E_2)] = 0$ .

### 3.1 R-matrix geometry

Here we shall investigate the geometrical properties associated to the R-matrix. As discussed in the introduction the first step is to obtain the defining equations for the corresponding variety  $Y$ . This requires the elimination of the weights  $\mathbf{x}_j, \mathbf{y}_j$  and  $\mathbf{z}_j$  from the ideal (4) built out of the bi-homogenous polynomials (15). The technicalities concerning this task are somehow similar to the elimination problem solved recently for the undeformed case [4]. Making the due adaptations we find that  $Y \in \mathbb{CP}^9$  is described as the intersection of four quadrics and one quartic polynomial,

$$Y = \{(\mathbf{a} : \mathbf{b} : \bar{\mathbf{b}} : \mathbf{c} : \bar{\mathbf{c}} : \mathbf{d} : \bar{\mathbf{d}} : \mathbf{f} : \mathbf{g} : \bar{\mathbf{g}}) \in \mathbb{CP}^9 \mid Q_1 = Q_2 = Q_3 = Q_4 = Q_5 = 0\}, \quad (24)$$

where the expressions for the polynomials  $Q_j$  are,

$$\begin{aligned} Q_1 &= \mathbf{b}\bar{\mathbf{b}} + \mathbf{a}\mathbf{g} - \mathbf{c}\bar{\mathbf{c}}, \quad Q_2 = \bar{\mathbf{b}}\mathbf{b} + \mathbf{f}\bar{\mathbf{g}} + q\mathbf{d}\bar{\mathbf{d}}, \\ Q_3 &= \mathbf{g}\mathbf{f} + \mathbf{a}\bar{\mathbf{g}} + (q + 1/q)\mathbf{b}\bar{\mathbf{b}}, \quad Q_4 = \mathbf{a}\mathbf{f} + \mathbf{g}\bar{\mathbf{g}} - (\mathbf{b}^2 + \bar{\mathbf{b}}^2), \\ Q_5 &= [(\bar{\mathbf{g}} - q\mathbf{g})(\mathbf{g} - q\bar{\mathbf{g}}) - (\mathbf{f} - q\mathbf{a})(\mathbf{a} - q\mathbf{f})]^2 - q^2\mathbf{U}^2\mathbf{c}\bar{\mathbf{c}}\mathbf{d}\bar{\mathbf{d}}. \end{aligned} \quad (25)$$

It turns out that  $Y$  is a four-dimensional complex algebraic variety such that part of its geometry is dominated by a two-dimensional projective space. This fact can be understood by noticing that in the subspace  $\mathbb{C}[\mathbf{c}, \bar{\mathbf{c}}, \mathbf{d}, \bar{\mathbf{d}}]$  the polynomials contain only monomials of the form  $\mathbf{c}\bar{\mathbf{c}}$  and  $\mathbf{d}\bar{\mathbf{d}}$ . They can be linearly eliminated with the help of the first two quadrics and the remaining equations become defined on the complementary ring  $\mathbb{C}[\mathbf{a}, \mathbf{b}, \bar{\mathbf{b}}, \mathbf{f}, \mathbf{g}, \bar{\mathbf{g}}]$ . This means that  $Y$  is birational to the product  $\mathbb{CP}^1 \times \mathbb{CP}^1 \times A$  where  $A$  is a surface defined by three polynomials, namely

$$A = \{(\mathbf{a} : \mathbf{b} : \bar{\mathbf{b}} : \mathbf{f} : \mathbf{g} : \bar{\mathbf{g}}) \in \mathbb{CP}^5 \mid Q_3 = Q_4 = \tilde{Q}_5 = 0\}, \quad (26)$$

where the expression for  $\tilde{Q}_5$  is,

$$\tilde{Q}_5 = [(\bar{\mathbf{g}} - q\mathbf{g})(\mathbf{g} - q\bar{\mathbf{g}}) - (\mathbf{f} - q\mathbf{a})(\mathbf{a} - q\mathbf{f})]^2 + q\mathbf{U}^2(\mathbf{b}\bar{\mathbf{b}} + \mathbf{a}\mathbf{g})(\bar{\mathbf{b}}\mathbf{b} + \mathbf{f}\bar{\mathbf{g}}). \quad (27)$$

At this point it remains to understand the geometric properties of the surface  $A$ . This investigation involves some technical steps summarized in Appendix B and in what follows we present the main conclusions. We first observe that by extracting the monomial  $\mathbf{b}\bar{\mathbf{b}}$  from third quadric the polynomial  $\tilde{Q}_5$  defines a surface on the ring subspace  $\mathbb{C}[\mathbf{a}, \mathbf{f}, \mathbf{g}, \bar{\mathbf{g}}]$ . The analysis of the geometry of such surface reveals that it is birational to a surface ruled by an elliptic curve isomorphic to  $E_2$ . The next step consists on the study of the normalization of the curve defined by the remaining polynomial  $Q_4$ . As a result we obtain that it is another genus one curve whose Weierstrass form is,

$$\begin{aligned} E_3 &= y^3 - x^3 + \frac{[16 + 8q(120q - \mathbf{U}^2)(1 + q^4) + q^2\mathbf{U}^4 - 240q^3\mathbf{U}^2 + 2144q^4 + 16q^8]}{48}x \\ &+ \frac{[4 + 4q^4 + 24q^2 - q\mathbf{U}^2][16 - 8q(264q + \mathbf{U}^2)(1 + q^4) + q^2\mathbf{U}^4 + 528q^3\mathbf{U}^2 - 4000q^4 + 16q^8]}{864}. \end{aligned} \quad (28)$$

Collecting the above information together we are able to conclude that  $A$  is birational to an Abelian surface determined by the product of two elliptic curves,

$$A \cong \bar{E}_2 \times \bar{E}_3. \quad (29)$$

We would like to close this section emphasizing that our conclusions for the geometric content are valid for generic points of the two-dimensional space of the couplings  $q$  and  $U$ . When these parameters are constrained to certain subspaces the respective varieties become reducible and this changes the geometrical properties. Potential geometric degenerations in the context of elliptic curves occur at the singularities of their  $J$ -invariants. Inspecting Eqs.(22) we see that this happens when,

$$qU^2 - 4(q^2 + \varepsilon)^2 = 0 \quad \text{with } \varepsilon = \pm 1. \quad (30)$$

In fact, under the condition (30) the sextic surface (13) becomes reducible in terms of the product of two cubic surfaces. The expressions of the irreducible components  $\bar{S}_{\pm}$  are,

$$\bar{S}_{\pm} = x^3 \pm x^2y/\sqrt{q} - qxy^2 \mp \sqrt{q}y^3 \pm \varepsilon xzw - q^{3/2}yzw, \quad (31)$$

and now the weights sit on a rational manifold since irreducible cubic surfaces are known to be birationally isomorphic to  $\mathbb{CP}^2$  [7].

Similar scenario is also expected for the geometry underlying the  $R$ -matrix and one direct way to see such decomposition is through a three-dimensional embedding of the surface  $A$ . This can be done by using the first four quadrics of Eq.(25) to eliminate in a linear way the variables  $\bar{\mathbf{c}}, \bar{\mathbf{d}}, \mathbf{f}$  and  $\bar{\mathbf{g}}$ . The remaining quartic  $Q_5$  gives rise to the surface  $A \in [\mathbf{a} : \mathbf{b} : \bar{\mathbf{b}} : \mathbf{g}]$  whose defining polynomial can be written as,

$$A = F_1^2 - \frac{U^2}{q}(\mathbf{a}\mathbf{g} + \mathbf{b}\bar{\mathbf{b}})F_2, \quad (32)$$

where the polynomials  $F_1$  and  $F_2$  are given by,

$$\begin{aligned} F_1 &= (\mathbf{a}^2 - \mathbf{g}^2)^2 + \mathbf{b}^4 + \bar{\mathbf{b}}^4 - 4\mathbf{a}\mathbf{b}\bar{\mathbf{b}}\mathbf{g} - (q + \frac{1}{q})(\mathbf{a}^2 + \mathbf{g}^2)(\mathbf{b}^2 + \bar{\mathbf{b}}^2) - (q^2 + \frac{1}{q^2})(2\mathbf{a}\mathbf{g} + \mathbf{b}\bar{\mathbf{b}})\mathbf{b}\bar{\mathbf{b}}, \\ F_2 &= (q + \frac{1}{q})(\mathbf{a}^2 + \mathbf{g}^2)(\mathbf{b}^2 + \bar{\mathbf{b}}^2)\mathbf{b}\bar{\mathbf{b}} + \left[ \mathbf{b}^4 + \bar{\mathbf{b}}^4 + (4 + q^2 + \frac{1}{q^2})\mathbf{b}^2\bar{\mathbf{b}}^2 \right] \mathbf{a}\mathbf{g} - \mathbf{b}\bar{\mathbf{b}}(\mathbf{a}^2 - \mathbf{g}^2)^2. \end{aligned} \quad (33)$$

We have verified that on the subspace (30) of couplings the octic surface defined by Eqs.(32,33) decomposes into lower degree polynomials. More specifically, we find that for  $\varepsilon = 1$  such reducibility is in terms of the product of two quartics surfaces while for  $\varepsilon = -1$  we have a product of four quadrics surfaces. In both cases all these surfaces components are birational to  $\mathbb{CP}^2$  and thus rational varieties.

Interesting enough, we also see that there is another simple degeneration possibility once we set  $U = 0$ . In this case the surface  $A$  becomes a square of the polynomial  $F_1$  for arbitrary values of  $q$ . It turns out that the quartic surface defined by  $F_1$  contains only simple singularities whose minimal resolution is known to be a K3 surface [7].

## 4 The symmetric gauge geometry

It has been noted in [1] that for a particular choice of the gauge parameter  $\gamma$  many of the off-diagonal amplitudes of the R-matrix becomes symmetric under transposition. This occurs when,

$$\gamma^2 \sim \frac{\sqrt{q(\xi + x^+)}(x^+ - x^-)}{\sqrt{(1 + \xi^2)(\xi + x^-)}}. \quad (34)$$

By substituting the relations (12) we conclude that Eq.(34) is equivalent to the condition that the ratio  $z/w$  is constant. Clearly, this plane does not intersect the ruled surface  $S$  on its  $\mathbb{CP}^1$  bundle and consequently the geometric properties of the Boltzmann weights may not be described by an elliptic curve. Without loss of generality we can set  $w = z$  for the symmetric gauge and the weights are now sited in the following curve,

$$\overline{C} = (x^2 - \frac{y^2}{q})(x^2 - qy^2)^2 - Uxyz^2(x^2 - qy^2) - z^4(x^2 - q^3y^2). \quad (35)$$

The above curve has three singular points one of them is an ordinary singularity while the others behave as tacnodes. The latter singularities behave like double point with only two tangent but having two branches, see for example [8]. The geometric genus  $g(\overline{C})$  is computed considering such singularities deficiencies and the result is,

$$g(\overline{C}) = \frac{5 \times 4}{2} - 1 - 2 \times 2 = 5. \quad (36)$$

In this situation the relationship among the curve  $\overline{C}$  with the original torus  $E_1$  is more severe than an isogeny. Indeed, the relation among such curves coordinates on the affine plane  $z = 1$  becomes,

$$x_+ = -\xi - \frac{\sqrt{1+\xi^2}}{\sqrt{q}} \left(\frac{y}{x}\right) (x^2 - qy^2), \quad x_- = -\xi - \frac{\sqrt{1+\xi^2}}{q^{3/2}} \left(\frac{x}{y}\right) (x^2 - qy^2), \quad (37)$$

given rise to a ramified mapping among curves explaining the drastic change on the genus. For some potential geometric degeneracies associated to the symmetric gauge see Appendix C.

Let us now turn our attention to the geometry properties of the R-matrix in the symmetric gauge. The expressions for the respective matrix elements are obtained setting  $\mathbf{z}_j = 1$  in the previous relations (15). We observe that we have two less independent matrix elements because of the identities  $\overline{\mathbf{c}} = \mathbf{c}$  and  $\overline{\mathbf{d}} = \mathbf{d}$ . Considering this fact it follows from Eqs.(25) that the underlying variety  $Z$  is now defined by,

$$Z = \{(\mathbf{a} : \mathbf{b} : \overline{\mathbf{b}} : \mathbf{c} : \mathbf{d} : \mathbf{f} : \mathbf{g} : \overline{\mathbf{g}}) \in \mathbb{CP}^7 \mid \overline{Q}_1 = \overline{Q}_2 = \overline{Q}_3 = \overline{Q}_4 = \overline{Q}_5 = 0\}, \quad (38)$$

where the polynomials  $\overline{Q}_j$  are only quadrics of the form,

$$\begin{aligned} \overline{Q}_1 &= \mathbf{b}\overline{\mathbf{b}} + \mathbf{a}\mathbf{g} - \mathbf{c}^2, \quad \overline{Q}_2 = \overline{\mathbf{b}}\mathbf{b} + \mathbf{f}\overline{\mathbf{g}} + q\mathbf{d}^2, \\ \overline{Q}_3 &= \mathbf{g}\mathbf{f} + \mathbf{a}\overline{\mathbf{g}} + (q + 1/q)\mathbf{b}\overline{\mathbf{b}}, \quad \overline{Q}_4 = \mathbf{a}\mathbf{f} + \mathbf{g}\overline{\mathbf{g}} - (\mathbf{b}^2 + \overline{\mathbf{b}}^2), \\ \overline{Q}_5 &= (\overline{\mathbf{g}} - q\mathbf{g})(\mathbf{g} - q\overline{\mathbf{g}}) - (\mathbf{f} - q\mathbf{a})(\mathbf{a} - q\mathbf{f}) - q\mathbf{U}\mathbf{c}\mathbf{d}. \end{aligned} \quad (39)$$

The algebraic set  $Z$  is a complete intersection and consequently we are dealing with a complex surface. One way to unveil its geometrical invariants is through a mapping to another variety whose geometric data is known. Here we are fortunate of being able to establish a simple map to the Abelian surface of previous section. Considering the  $\mathbb{CP}^3$  embeddings for  $A$  and  $Z$  given by Eqs.(32,33,C.2,C.3) we can set the mapping,

$$\begin{aligned} Z \subset \mathbb{C}[\mathbf{a}, \mathbf{b}, \overline{\mathbf{b}}, \mathbf{c}] & \xrightarrow{\psi} A \subset \mathbb{C}[\mathbf{a}, \mathbf{b}, \overline{\mathbf{b}}, \mathbf{g}] \\ [\mathbf{a} : \mathbf{b} : \overline{\mathbf{b}} : \mathbf{c}] & \longmapsto [\mathbf{a}^2 : \mathbf{a}\mathbf{b} : \mathbf{a}\overline{\mathbf{b}} : \mathbf{c}^2 - \mathbf{b}\overline{\mathbf{b}}]. \end{aligned} \quad (40)$$



The map  $\psi$  has degree two being regular in the open set  $\mathbf{a} = 1$  containing four ramification lines at  $\mathbf{c} = 0$ . From Hironaka desingularization theorem [9] it follows that by a succession of monoidal transformations we can eliminate the indeterminacy locus of  $\psi$  resulting in a morphism,

$$\tilde{Z} \subset \mathbb{CP}^3 \xrightarrow{\tilde{\psi}} \tilde{A} \subset \mathbb{CP}^3, \quad (41)$$

connecting the birational models  $\tilde{Z}$  and  $\tilde{A}$  of the surfaces  $Z$  and  $A$ , respectively.

Now the map  $\tilde{\psi}$  defines a double covering branched along the union of disjoint smooth curves with an effective locus say  $B \in \tilde{A}$ . Since the work by Persson [10] it is known that a smooth double cover of the surface  $\tilde{A}$  is uniquely determined by a line bundle  $\mathcal{L}$  on  $\tilde{A}$  such that  $B \in |2\mathcal{L}|$ . For recent overview on the properties of the invariants of double coverings of surfaces see for instance [11]. It turns out that from this construction we can uncover the geometric data of  $\tilde{Z}$  as follows,

- The Euler-Poincaré characteristic  $\chi(S)$  of surface  $S$

$$\chi(\tilde{Z}) = 2\chi(\tilde{A}) + \frac{1}{2} (\mathcal{L}, \mathcal{L} + K_{\tilde{A}}) = \frac{1}{2} (\mathcal{L}, \mathcal{L}), \quad (42)$$

where  $K_S$  is the canonical bundle of  $S$  and  $(\mathcal{L}, D)$  denotes the intersection number of the line bundle and a divisor  $D \in \tilde{A}$ .

- The self-intersection number  $K_S^2$  on the surface  $S$

$$K_{\tilde{Z}}^2 = 2K_{\tilde{A}}^2 + 2 (\mathcal{L}, \mathcal{L} + K_{\tilde{A}}) + 2 (\mathcal{L}, K_{\tilde{A}}) = 2 (\mathcal{L}, \mathcal{L}). \quad (43)$$

- The geometric genus  $p_g(S)$  of the surface  $S$

$$p_g(\tilde{Z}) = p_g(\tilde{A}) + \dim_{\mathbb{C}} H^0(\tilde{A}, K_{\tilde{A}} \otimes \mathcal{L}) = 1 + \dim_{\mathbb{C}} H^0(\tilde{A}, \mathcal{L}) \quad (44)$$

where  $H^i(\tilde{A}, \mathcal{L})$  denotes the  $i$ -th cohomology group of the sheaf associated to the line bundle  $\mathcal{L}$ .

- The irregularity  $q(S)$  of the surface  $S$

$$q(\tilde{Z}) = 1 + p_q(\tilde{Z}) - \chi(\tilde{Z}) = 2 + \dim_{\mathbb{C}} H^0(\tilde{A}, \mathcal{L}) - \frac{1}{2} (\mathcal{L}, \mathcal{L}) \quad (45)$$

The last equality in the above formulas considered the properties of an Abelian surface, that is  $\chi(\tilde{A}) = 0$  and  $p_q(\tilde{A}) = 1$  as well as the fact  $K_{\tilde{A}}$  is trivial. At this point we see from Eqs.(42,43) that the corresponding invariants satisfy the relation,

$$K_{\tilde{Z}}^2 = 4\chi(\tilde{Z}) \quad (46)$$

which means that the surface  $\tilde{Z}$  sits on the so-called Severi line [5].

From now on we assume that  $\mathcal{L}$  is an ample line bundle and under this mild hypothesis we can relate the dimension of  $H^0(\tilde{A}, \mathcal{L})$  with the self-intersection number of the line bundle. Indeed, from the Riemann-Roch theorem [7] for the Euler-Poincaré characteristic of  $\mathcal{L}$  we have,

$$\dim_{\mathbb{C}} H^0(\tilde{A}, \mathcal{L}) - \dim_{\mathbb{C}} H^1(\tilde{A}, \mathcal{L}) + \dim_{\mathbb{C}} H^2(\tilde{A}, \mathcal{L}) = \frac{1}{2} (\mathcal{L}, \mathcal{L}) \quad (47)$$

and considering that the ampleness assumption for the line bundle implies  $\dim_{\mathbb{C}} H^1(\tilde{A}, \mathcal{L}) = \dim_{\mathbb{C}} H^2(\tilde{A}, \mathcal{L}) = 0$  we obtain from Eq.(47) the simple relation,

$$\dim_{\mathbb{C}} H^0(\tilde{A}, \mathcal{L}) = \frac{1}{2} (\mathcal{L}, \mathcal{L}) \quad (48)$$

Note that the above relation together with Eq.(45) already fix the irregularity of  $\tilde{Z}$ , namely

$$q(\tilde{Z}) = 2 \quad (49)$$

The complete understanding of the geometric properties of  $\tilde{Z}$  still needs the knowledge of the line bundle intersection number. We can retrieve this number with the help of the adjunction formula for a generic curve  $\tilde{C}$  on  $\tilde{A}$ , namely

$$g(\tilde{C}) = 1 + \frac{1}{2} (\tilde{C}, \tilde{C}) \quad (50)$$

We stress here that the self-intersection number for  $\tilde{C}$  is defined to be the intersecting number of the corresponding line bundle. Hence for any curve  $\tilde{C}$  in the linear system of the ample line bundle  $\mathcal{L}$  we can reverse Eq.(50) and state that,

$$(\mathcal{L}, \mathcal{L}) = 2[g(\tilde{C}) - 1] \quad (51)$$

We now are left to compute the genus of an ample divisor on the original Abelian surface such as the intersection of  $A$  with a hyperplane in some projective embedding. For this purpose it is suffice to take the curve obtained from Eq.(32,33) when we set for instance  $\mathbf{g} = 0$ ,

$$\begin{aligned} C = & \left[ \mathbf{a}^4 + \mathbf{b}^4 + \overline{\mathbf{b}}^4 - (q + \frac{1}{q})\mathbf{a}^2(\mathbf{b}^2 + \overline{\mathbf{b}}^2) - (q^2 + \frac{1}{q^2})(\mathbf{b}\overline{\mathbf{b}})^2 \right]^2 \\ & - \frac{U^2}{q}(\mathbf{a}\mathbf{b}\overline{\mathbf{b}})^2 \left[ (q + \frac{1}{q})(\mathbf{b}^2 + \overline{\mathbf{b}}^2) - \mathbf{a}^2 \right] \end{aligned} \quad (52)$$

It turns out that such degree eight plane curve has twelve ordinary singularities and all of them have the multiplicity index of double points. The genus of its desingularization  $\tilde{C}$  is therefore easily computed as,

$$g(\tilde{C}) = \frac{7 \times 6}{2} - 12 = 9 \quad (53)$$

and consequently we obtain  $(\mathcal{L}, \mathcal{L}) = 16$ .

Now, collecting together all the above information we conclude that the surface we have started with is of general type whose main geometric data is,

$$q(\tilde{Z}) = 2 \quad \text{and} \quad p_q(\tilde{Z}) = 9 \quad (54)$$

In addition to that, considering the values  $\chi(\tilde{Z}) = 8$  and  $K_Z^2 = 32$  we can predict the behaviour of the corresponding higher  $n$ -plurigena, namely

$$P_n(\tilde{Z}) = \chi(\tilde{Z}) + \frac{n(n-1)}{2}K_Z^2 = 8[1 + 2n(n-1)] \quad \text{for } n > 2 \quad (55)$$

Finally, we remark that we have checked some of these numerical values within the formal surface desingularization routine implemented in the computer algebra system Magma [12]. Not only we have been able to confirm the values for irregularity and the geometric genus but also the first two  $P_2(\tilde{Z})$  and  $P_3(\tilde{Z})$  plurigena numbers.

## 5 Conclusions

In this work, we have derived a formulation for the R-matrix based on a  $q$ -deformation of the centrally extended  $sl(2|2)$  superalgebra towards the view of algebraic geometry. This made it pos-

sible to uncover the geometric properties of the elementary weights as well as of the corresponding R-matrix.

Our analysis made clear that if  $X$  denotes the variety associated to the elementary weights the geometry underlying the R-matrix is not necessarily described by the product  $X \times X$ . We have argued that this change is rather drastic for the symmetric choice of the gauge parameter. In fact, in this case  $X$  is the genus five curve  $\overline{C}$  (35) and the geometric data of the product  $\overline{C} \times \overline{C}$  can be retrieved directly from the genus of the respective curve. Though this also gives rise to a surface of general type the geometrical invariants are,

$$q(\overline{C} \times \overline{C}) = 5 + 5 = 10 \quad \text{and} \quad p_g(\overline{C} \times \overline{C}) = 5 \times 5 = 25 \quad (56)$$

which are quite distinct from the actual geometrical content associated to the R-matrix, see Eq.(54).

We have observed that the geometric properties can change to other classes of universalities when the couplings are restricted to the subspace (30). It is reasonable to think that such geometry change will reflect on an equivalent modification of the physical properties associated to the respective vertex model and spin chain. It seems worthwhile to investigate the way the geometric properties may be encoded for instance on the nature of the excitations of the spin chain. It seems also of interest to carry on the algebraic Bethe ansatz for both the eigenvectors and eigenvalues of the transfer matrix. We expect that the derived identities (39) among the R-matrix entries will be useful for such algebraic formulation.

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## Appendix A: Uniformization of $E_2$

We note that the curve  $E_2$  can be rewritten as,

$$-\frac{y_1^2}{4q} = (1 - \lambda_1 y_2^2)(1 - \lambda_2 y_2^2), \quad (\text{A.1})$$

where the parameters  $\lambda_1$  and  $\lambda_2$  satisfy the following relations,

$$\lambda_1 + \lambda_2 = \frac{4 + 4q^4 - qU^2}{4q}, \quad \lambda_1 \lambda_2 = q^2. \quad (\text{A.2})$$

Rescaling the variables  $y_1 \rightarrow 2i\sqrt{q}y_1$  and  $y_2 \rightarrow y_2/\sqrt{\lambda_1}$  we can bring Eq.(A.1) in the standard Jacobi form. A natural uniformization is therefore in terms of the Jacobi's elliptic functions, namely

$$y_1 = 2i\sqrt{q}\text{cn}(\mu, \mathbf{k})\text{dn}(\mu, \mathbf{k}), \quad y_2 = \sqrt{\frac{\mathbf{k}}{q}}\text{sn}(\mu, \mathbf{k}), \quad (\text{A.3})$$

where  $\mu$  is a spectral parameter and  $\mathbf{k}$  is the modulus of the elliptic functions. The latter is given in terms of the couplings  $q$  and  $U$  by,

$$\mathbf{k} = \frac{\Delta}{2} \pm \sqrt{\Delta^2/4 - 1}, \quad \Delta = q^2 + \frac{1}{q^2} - \frac{U^2}{4q}. \quad (\text{A.4})$$

## Appendix B: Surface Analysis

After extracting the product  $\mathbf{b}\bar{\mathbf{b}}$  from quadric  $Q_3$  and substituting it in Eq.(27) we obtain,

$$\begin{aligned} \tilde{Q}_5 &= [(\bar{\mathbf{g}} - q\mathbf{g})(\mathbf{g} - q\bar{\mathbf{g}}) - (\mathbf{f} - q\mathbf{a})(\mathbf{a} - q\mathbf{f})]^2 \\ &+ \frac{q^3 U^2}{(1 + q^2)^2} [\mathbf{f}(q\bar{\mathbf{g}} - \mathbf{g}) + \bar{\mathbf{g}}(\mathbf{f}/q - \mathbf{a})] [\mathbf{a}(q\mathbf{g} - \bar{\mathbf{g}}) - \mathbf{g}(\mathbf{f} - \mathbf{a}/q)]. \end{aligned} \quad (\text{B.1})$$

The geometry of the above quartic surface can be understood by means of composition of birational transformations. We start by defining the following auxiliary variables,

$$\mathbf{h} = \mathbf{a} - \mathbf{f}/q, \quad \bar{\mathbf{h}} = \mathbf{a} - q\mathbf{f}, \quad \mathbf{p} = \mathbf{g} - \bar{\mathbf{g}}/q, \quad \bar{\mathbf{p}} = \mathbf{g} - q\bar{\mathbf{g}}. \quad (\text{B.2})$$

We then observe that  $\tilde{Q}_5$  becomes quadratic in the variable  $\bar{\mathbf{p}}$  and the linear term can be eliminated by quadrature. More precisely, we are able to perform the following transformation,

$$\frac{\bar{\mathbf{p}}}{\mathbf{h}} = \frac{i\sqrt{q}(q^4 - 1)Ux_0x_1 - [q(1 + q^4)U^2 - 2(1 - q^4)^2]\mathbf{p}\bar{\mathbf{h}}}{2[(q^4 - 1)^2\mathbf{p}^2 - qU^2\bar{\mathbf{h}}]}, \quad (\text{B.3})$$

replacing the variables  $\bar{\mathbf{p}}$  and  $\mathbf{h}$  by the new homogeneous coordinates  $x_0$  and  $x_1$ .

As a result we find that the expression of the surface  $\tilde{Q}_5$  in terms of these new variables is,

$$\tilde{Q}_5 = x_0^2x_1^2 + 4q^4\mathbf{p}^4 - (4 - qU^2 + 4q^4)\mathbf{p}^2\bar{\mathbf{h}} + 4\bar{\mathbf{h}}^4, \quad (\text{B.4})$$

which is exactly the ruled surface (13) upon re-scaling of the coordinates  $\mathbf{p} = \frac{x_2}{q^{3/4}}$  and  $\bar{\mathbf{h}} = q^{3/4}x_3$ .

The last step in the analysis concerns with the study of the curve originated from the polynomial  $Q_4$ . Taking into account the above information we find that it has the following structure,

$$\begin{aligned} Q_4 = & \mathbf{b}^4 + P_1(y_1, y_2)\mathbf{b}^2x_2^2 + P_2(y_1, y_2)\mathbf{b}^2x_2 + P_3(y_1, y_2)\mathbf{b} + P_4(y_1, y_2)x_2^4 + P_5(y_1, y_2)x_2^3 \\ & + P_6(y_1, y_2)x_2^2 + P_7(y_1, y_2)x_2 + P_8(y_1, y_2), \end{aligned} \quad (\text{B.5})$$

where the coefficients  $P_j(y_1, y_2)$  belong to the field of fractions of  $E_2$ .

We find that this curve has two ordinary double points as singularities and therefore its normalization  $E_3$  is an elliptic curve. Remarkably, the respective J-invariant does depend on the curve  $E_2$  and its explicit value is,

$$J(E_3) = \frac{(16 - 8qU^2 + q^2U^4 + 960q^2 - 240q^3U^2 + 2144q^4 - 8q^5U^2 + 960q^6 + 16q^8)^3}{q^2(4 - qU^2 + 4q^4 + 8q^2)(4 - qU^2 + 4q^4 - 8q^2)^4}. \quad (\text{B.6})$$

## Appendix C: Symmetric Gauge Degenerations

In this case the genus five curve (35) decomposes into two irreducible and isomorphic elliptic curves provided the constrain (30) is satisfied. The expression of one of the components is,

$$\bar{C} = x^3 + x^2y/\sqrt{q} - qxy^2 - \varepsilon\sqrt{q}y^3 + \varepsilon xz^2 - q^{3/2}yz^2. \quad (\text{C.1})$$

Interesting enough, we note that the the two possible cases for  $\varepsilon$  are distinguished by the J-invariant,

$$J(\overline{C}) = \begin{cases} \frac{64(q^2+3)^3(3q^2+1)^3}{(q^2-1)^4(q^2+1)^2} & \text{for } \varepsilon = 1 \\ 1728 & \text{for } \varepsilon = -1. \end{cases}$$

As in the main text we can embed the variety  $Z$  in a  $\mathbb{CP}^3$  projective space. For  $U \neq 0$  we can use the quadrics (39) to eliminate the elements  $\mathbf{d}, \mathbf{f}, \mathbf{g}, \overline{\mathbf{g}}$  and such embedding is given by,

$$Z = F_3^2 - \frac{U^2}{q} \mathbf{a}^4 \mathbf{c}^2 F_4, \quad (\text{C.2})$$

where the polynomials  $F_3$  and  $F_4$  are,

$$\begin{aligned} F_3 &= (\mathbf{a}^2 + \mathbf{c}^2 - \mathbf{b}\overline{\mathbf{b}})^2 (\mathbf{a}^2 - \mathbf{c}^2 + \mathbf{b}\overline{\mathbf{b}})^2 + \mathbf{a}^4 [\mathbf{b}^4 + \overline{\mathbf{b}}^4 - 4\mathbf{b}\overline{\mathbf{b}}(\mathbf{c}^2 - \mathbf{b}\overline{\mathbf{b}})] \\ &\quad - (q + \frac{1}{q}) \mathbf{a}^2 (\mathbf{b}^2 + \overline{\mathbf{b}}^2) [\mathbf{a}^4 + (\mathbf{c}^2 - \mathbf{b}\overline{\mathbf{b}})^2] + (q^2 + 1/q^2) \mathbf{a}^4 \mathbf{b}\overline{\mathbf{b}} (\mathbf{b}\overline{\mathbf{b}} - 2\mathbf{c}^2), \\ F_4 &= (q + \frac{1}{q}) \mathbf{a}^2 \mathbf{b}\overline{\mathbf{b}} (\mathbf{b}^2 + \overline{\mathbf{b}}^2) [\mathbf{a}^4 + (\mathbf{c}^2 - \mathbf{b}\overline{\mathbf{b}})^2] + (4 + q^2 + \frac{1}{q^2}) \mathbf{a}^4 \mathbf{b}^2 \overline{\mathbf{b}}^2 [\mathbf{c}^2 - \mathbf{b}\overline{\mathbf{b}}] \\ &\quad - \mathbf{b}\overline{\mathbf{b}} (\mathbf{a}^2 + \mathbf{c}^2 - \mathbf{b}\overline{\mathbf{b}})^2 (\mathbf{a}^2 - \mathbf{c}^2 + \mathbf{b}\overline{\mathbf{b}})^2 + \mathbf{a}^4 (\mathbf{b}^4 + \overline{\mathbf{b}}^4) (\mathbf{c}^2 - \mathbf{b}\overline{\mathbf{b}}). \end{aligned} \quad (\text{C.3})$$

For  $\varepsilon = 1$  the surface (C.2,C.3) factorizes into the product of two octic surfaces which can be seen as a ramified double cover over  $\mathbb{CP}^2$  and thus their normalization are K3 surfaces. On the other hand for  $\varepsilon = -1$  we have a factorization on the complex field in terms of eight quadrics and consequently  $Z$  is described by rational surfaces.

## References

- [1] N. Beisert and P. Koroteev *J.Phys.A:Math.Theor.* *41* (2008) 255204
- [2] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, New York, 1982.
- [3] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties and Algorithms*, 3rd edition, Springer, New York, 2007.
- [4] M.J. Martins, *Nucl.Phys.B* *907* (2016) 479

- [5] F. Severi, *Comm.Math.Helv.* 4 (1932) 268; M. Manetti, *Math.Nachr.* 261 (2003) 105
- [6] J.H. Silverman, *The Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, Vol 106, Springer-Verlag, 1986.
- [7] A. Beauville, *Complex Algebraic Surfaces* , London Mathematical Society Students Text, Vol 34, Cambridge University Press, New York, 1996.
- [8] W.J. Walker, *Algebraic Curves*, Dover Publications, 1950.
- [9] H. Hironaka, *Ann. of Math.* 79 (1964) ,II,109
- [10] U. Persson, *Compos. Math.* 43 (1981) 3
- [11] W.P. Barth, K. Hulek, C.A.M. Peters and A. van de Ven *Compact Complex Surfaces* ,  
Ergebn.der Mathem., Vol 4, Springer, Berlin, 2003.
- [12] C. Bosma, J. Cannon and C. Playoust, *The Magma Algebra System* , Computational algebra  
and number theory, V2-22-3, <http://magma.maths.usyd.edu.au>